Framework for Mathematical Proficiency for Teaching

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During the last few years, we and our colleagues at Penn State have been attempting to design a framework for the construct of *mathematical knowledge for teaching* (MKT) as it might be applied to secondary school mathematics. Working from the bottom up, we began by developing a collection of sample situations. Each situation portrays an incident in teaching secondary mathematics in which some mathematical point is at issue. (For details of our approach, see Kilpatrick, Blume, & Allen, 2006.) Using the situations, we have attempted to identify the special knowledge of secondary school mathematics that teachers should have but that other users of mathematics would not necessarily need. Looking across situations, we have tried to characterize that knowledge.

Our initial characterization was much influenced by the work of Deborah Ball and her colleagues at the University of Michigan (Ball, 2003; Ball & Bass, 2000; Ball, Bass, & Hill, 2004; Ball, Bass, Sleep, & Thames, 2005; Ball & Sleep, 2007). In particular, Ball et al. have partitioned MKT into components that distinguish between subject matter knowledge and pedagogical content knowledge (Shulman, 1986). Early in their work, they identified four components: common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching (Ball et al., 2004). And more recently, they have added two additional kinds of knowledge: knowledge of curriculum and knowledge at the mathematical horizon. An example of the latter is "being aware that two-digit multiplication anticipates the more general case of binomial multiplication later in a student's mathematical career" (Ball, 2003, p. 4). Figure 1 shows the six components and how they are related.

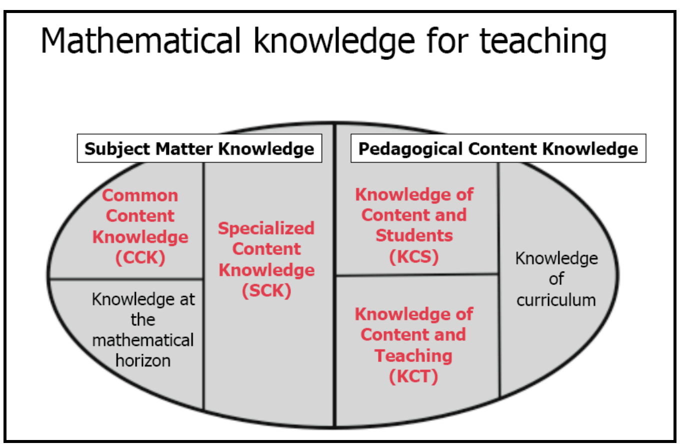

Figure 1. Model of MKT (Ball & Sleep, 2007).

As we worked on developing our own framework, we considered attempts to develop frameworks related to mathematical knowledge for teaching (e.g., Adler & Davis, 2006; Cuoco, 2001; Cuoco, Goldenberg, & Mark, 1996; Even, 1990; Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005; McEwen & Bull, 1991; Peressini, Borko, Romagnano, Knuth, & Willis-

Yorker, 2004; Tatto et al., 2008). We became increasingly concerned that whatever framework we developed, it needed to reflect a broader, more dynamic view of mathematical knowledge.

The philosopher Gilbert Ryle (1949) claimed that there are two types of knowledge: The first is expressed as "knowing that," sometimes called *propositional* or *factual* knowledge, and the second as "knowing how," sometimes called *practical* knowledge. We wanted to capture this distinction and at the same time to enlarge the MKT construct to include such aspects as reasoning, problem solving, and disposition. Consequently, we adopted the term *proficiency,* which we use in much the same way as the term is used in *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001[1]). We consider mathematical proficiency for teaching (MPT) to include the expertise, competence, knowledge, and facility in mathematics that is useful to teachers of mathematics at the secondary level. We are interested in both factual and practical mathematical knowledge that can be used in the work of teaching mathematics. We see the content dimension of MPT as comprising a number of strands that go beyond a simple contrast between knowledge and understanding. We also include a teaching dimension of MPT as a way of combining practical knowledge and factual knowledge in order to capture how teachers' mathematical proficiency is situated in their classroom practice. It should be understood that along either of the two dimensions, teachers' proficiency can be at any level of development from novice to expert. Our current framework is shown in Figure 2.

1. Mathematical Proficiency with Content
   - Conceptual understanding
   - Procedural fluency
   - Strategic competence
   - Adaptive reasoning
   - Productive disposition
   - Cultural and historical knowledge
   - Knowledge of structure and conventions
   - Knowledge of connections within and outside the subject

2. Mathematical Proficiency in Teaching
   - Knowing students as learners
   - Assessing one's teaching
   - Selecting or constructing examples and tasks
   - Understanding and translating across representations
   - Understanding and using classroom discourse
   - Knowing and using the curriculum
   - Knowing and using instructional tools and materials

Figure 2. Framework for mathematical proficiency for teaching.

[1] Recognizing that no term captures completely all aspects of expertise, competence, knowledge, and facility in mathematics, we have chosen mathematical proficiency to capture what we believe is necessary for anyone to learn mathematics successfully. (p. 115)

Elaboration

1. Mathematical Proficiency with Content

Conceptual understanding

To have conceptual understanding, a teacher must be aware of the characteristic details of a topic as well the relationships that link those details to each other and to other bits of information (Hiebert and Lefevre, 1986). In essence, conceptual understanding adds depth to a teacher's knowledge base. It requires more than the accumulation of facts by asking that the teacher understand how those facts can be laterally connected within the subject area as well as vertically linked to past and future content areas.

Procedural fluency

Whereas conceptual understanding focuses on connections, procedural fluency focuses on the specifics of the task at hand. In order to be procedurally fluent, a secondary mathematics teacher must understand both the form and the "how to" of completing tasks (Hiebert and Lefevre, 1986). Understanding the form of the mathematics requires that the teacher be familiar with appropriate notation as well as how the notation is used to articulate an argument. In addition to an understanding of form, the teacher must also possess an understanding of the processes required to complete a task. These processes can include the application of rules, the execution of algorithms, and the implementation of problem-solving strategies.

Strategic competence

Strategic competence is the "ability to formulate mathematical problems, represent them, and solve them" (Kilpatrick, Swafford, & Findell, 2001). Using multiple strategies, verifying conjectures, and working with representations are some of the techniques that may be grouped under strategic competence.

Adaptive reasoning

Adaptive reasoning includes the ability to reason both formally and informally as well as the tools and components of reasoning. Mathematically proficient teachers have a facility with different types of reasoning including deductive reasoning, inductive reasoning, reasoning from representations, reasoning through analogies, probabilistic reasoning, and justifying. Adaptive reasoning includes understanding the nature of definitions, formal logic, conjectures, examples and counter examples as well as how these tools are used to reason. Understanding necessary and sufficient conditions within an argument is part of adaptive reasoning. Understanding the processes of generalizing, proving, and refuting are part of reasoning. Proficiency in mathematical reasoning includes knowing productive questions, investigating special cases, classifying, and checking the validity of statements for different domains of numbers or elements. Adaptive reasoning includes informal approaches the build from questions, exploration, and plausible explanations, and often provide insights for more formal approaches such as arguing by contradiction, exhausting each possibility, deducing from assumptions, or deducing from definitions. Adaptive reasoning includes messy, lengthy work of proving that is often hidden behind products such as elegant proofs.

Productive disposition

An operating assumption of this framework is that increased mathematical proficiency is beneficial and possible, both for secondary students and for their teachers. A productive disposition for secondary teachers exists when those teachers share this assumption. More generally, teachers need effective mental practices that allow them to think efficiently about ideas. Some of these mental practices are specific to mathematics while others are productive "habits of mind" useful in all disciplines, including: pattern sniffing, experimenting, describing, tinkering, inventing, visualizing, conjecturing, guessing (Cuoco, Goldenberg, & Mark, 1996).

Cultural and historical knowledge

No matter what one's epistemological beliefs are about the nature of mathematics, mathematics is a human endeavor. From a historical viewpoint, many topics in secondary mathematics provide teachers with an opportunity for their students to explore much of that history. Non-Euclidean geometry may be used to promote a better model of the world on which the students are living. It can also be used to reinforce the deductive nature of mathematics since many of the theorems from Euclidean geometry look similar in some non-Euclidean geometries. Cultural knowledge helps a teacher connect mathematics to the contributions made by many societies. Eygptian, Babylonian, Hindu, and Arabic societies have all impacted the mathematics students study today.

Knowledge of structure and conventions

MPT at the secondary level differs from that of earlier levels in that the underlying structure of mathematical ideas gradually becomes more explicit. Secondary teachers ask their students to grasp this structure more fully in many ways, including by:

1. Extending operations
2. Exploring similarities between relationships, as well as between objects[2]
3. Making previous ideas more rigorous[3]

These advancements in mathematical understanding necessitate an advancement in mathematical communication and computation, which may include:

1. Making finer distinctions
2. Developing more elaborate notation
3. Developing more complex algorithms

To facilitate mathematical learning at the secondary level, it is essential that teachers be cognizant of the difference between the above two lists, and the possibility that this difference may not be clear to those learning the ideas from both lists somewhat simultaneously. The first list represents those ideas, knowledge of which is obtainable simply by logic, given time and impetus. The second list represents the customary and rather arbitrary ways and means that have been and are still being developed by people seeking to understand, utilize, and communicate about those ideas in the first list.

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[2] An example of this would be a recognition of the difference between functions and 1-1 functions.

[3] The difference between open and closed intervals is more carefully distinguished, for example.

Knowledge of connections within and outside the subject

The connections mathematics teachers can make *within* the subject include those across various branches of mathematics (geometry, algebra, statistics, etc.), as well as those between levels of mathematics: students' prior knowledge and what they are likely to study next. Connections can also be made to deeper mathematics—these could be called extensions. One interesting way to extend a problem or a concept is to alter the assumptions: for example, in a geometry problem one might consider non-Euclidean geometry. Connections *outside* the subject include such things as applications of mathematical concepts to the "real world" (that is, beginning with mathematical concepts, then finding instances of those concepts), or applications to other school subjects. Another kind of connection is mathematical modeling, in which one begins by investigating phenomena in the "real world," then draws out mathematical concepts seen there.

2. Mathematical Proficiency in Teaching

Knowing students as learners

Every student brings his or her own previous mathematical experiences into a mathematics classroom and each lesson provides the student an opportunity to grow their mathematical knowledge through a variety of learning tasks. A teacher who possesses MPT creates hypothetical learning trajectories for their students' mathematical understanding, building a bridge between previous understanding and new mathematical material (Simon, 1995). From this, the teacher has an idea of the students' level of understanding and plans instruction to increase the students' mathematical knowledge (NCTM, 2007).

In understanding the learner, teachers understand that the student brings his or her own culture to the classroom. Each culture addresses mathematics in different ways. Understanding how the student's culture contributes not only to the field of mathematics but how each student learns mathematics is a component of MPT. In order to understand the students as learners, students need to appreciate the entire person, including his or her culture.

Assessing one's teaching

There are a wide variety of ways to assess one's teaching. Most of these could be considered as forms of reflection on one's own work. Some of them, however, may involve more objective approaches, such as examining video records or working with another teacher, teacher-educator, or administrator to look at an individual's teaching. The particular framework we use to consider the art of teaching—static, dynamic, or decision-based—is irrelevant when it comes to the need for self-assessment. In all models it is vital for the teacher to review the work he or she has done and/or the results of that work, such as evidence of student learning. He can then evaluate it for effectiveness. Evaluation may be done at any point in the process of teaching, not just after a lesson has been completed (or taught). For instance, as a teacher is engaged in teaching a lesson, he or she should monitor his or her activity, including the reactions of the students, to see if the work is effective, *i.e.* there is evidence of student learning, and, if not, to consider an alternative course of action that might result in increased student understanding . Self-assessment, whatever form it takes, is one of many tools the teacher employs to be more aware of his or her practice. The information gathered during self-assessment should inform an understanding of a teacher's practice and allow him or her to continuously improve it.

Selecting or constructing examples and tasks

Selecting or constructing instructionally powerful examples is a common and important activity of teaching mathematics. One must come up with an example, non-example, or counterexample to address the concept at hand, without introducing unnecessary ambiguity. A teacher must also recognize the value or limitations of examples that others may introduce. Mathematical tasks are those that serve as opportunities for students to learn mathematical concepts. The teacher is concerned with selecting or creating tasks that are appropriate to the level of mathematics of the class, and also keeping high standards for the level of cognitive demand for students.

Understanding and translating across representations

Teachers use representations throughout the course of the day to help students learn. Students represent their work and their answers with words and pictures. Understanding and translating across representations has been a key element in many (some?) of our situations and foci. When students present solutions to tasks that are both correct and yet different, a teacher will be able to connect those two representations. When deciding upon representing a function in a lesson plan, a teacher will be able to choose the representation (or representations) which will have the best chance to illustrate the function. Connecting between a physical model, such as a line drawn on a board or a plastic Platonic solid, and their mathematical counterparts requires some finesse. We have found evidence in our situations that some representations generated by students, while seemingly strong models of the concept, present difficulties and possible barriers to the mathematical understanding of the concept.

Understanding and using classroom discourse

The intercommunication between and among students and teachers is vital. Classroom interactions play a significant role in teachers' understandings of what their students know and are learning. Examining classroom discourse can reveal how both students and teachers understand and make connections between the mathematical ideas being discussed. In order for this to happen more effectively, teachers can benefit from an understanding of discourse on both the theoretical and practical levels. Theoretical understanding guides the teachers' understanding of the importance of appropriate discourse practices. Reading and incorporating what is learned from research on discourse provides the teacher with additional information about incorporating discourse into practice. Building a practical understanding of, and knowledge base of actions for, engaging students in discourse about important mathematical ideas informs and guides teaching practice and enhances the impact and usefulness of the practice for teachers and learners alike.

Knowing and using the curriculum

How mathematical knowledge is used to teach mathematics in a specific classroom, or with a specific learner, or a specific group of learners is influenced by the curriculum that organizes the teaching and learning. A teacher's mathematical proficiency can make the curriculum meaningful, connected, relevant, and useful. For example, a teacher who is mathematically proficient can think of teaching the concept of area as part of a curriculum that includes ideas about measure, descriptions of two-dimensional space, measures of space under a curve, measures of the surface of three-dimensional solids, infinite sums of discrete regions, operations on space and measures of space, and useful applications involving area. This is a very different perspective of the curriculum from someone who thinks of area in terms of formulas for polygonal regions.

Mathematical proficiency for knowing and using the curriculum in teaching requires a teacher to identify foundational or prerequisite concepts that enhance the learning of a concept as well as how the concept being taught can serve as a foundational or requisite concept for future learning. A teacher needs to know how a particular concept fits within a student's learning trajectory. At the same time, proficient mathematics teachers understand that there is not prescribed, linear order for learning mathematics, but rather multiple mathematical ways to approach a concept and to revisit a concept. Mathematical concepts and processes evolve in the learner's mind becoming more complex and sophisticated with each iteration. Mathematical proficiency prepares a teacher to build a curriculum that not only connects mathematical ideas but builds a disposition within students where they expect mathematical ideas to be connected (Cuoco, 2001).

A mathematically proficient teacher understands that a curriculum contains not only mathematical entities but also mathematical processes for relating, connecting, and operating on those entities (NCTM Standards, 1989, 2000). A teachers must have mathematical proficiency to set appropriate curricular goals for their students (Adler, 2006). For example, a teacher needs mathematical knowledge to select and teach functions that help students build a basic repertoire of functions (Even, 1990).

Knowing and using instructional tools and materials

When determining the set of instructional tools that teachers might implement in their classroom, technological tools like graphing calculators and computer software comprise part of that list. The use of technology allows students to see many examples of a concept in a short period of time. In identifying MPT, teachers might notice where the student can begin to abstract their understanding from concrete examples into a conceptual understanding. Although teachers need not have a background in programming the technology, there is mathematics involved in the implementation of technology in the classroom. Instructional tools are not limited to technology. MPT can be found embedded in the choice of manipulatives that teachers use as part of their lessons. In selecting manipulatives and other visual aids, the teacher would need to address the mathematics in choosing certain manipulatives and the extensions that can be made when they are implemented as part of a daily lesson.

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